Problem 13

Solve the inequality $\ln(x^2 - 2x - 2) \leq 0$.

Solution

There are actually two inequalities to solve here; one of them is implicit and comes from the fact that the argument of a logarithm must be positive. The values of x we seek must satisfy

$$\begin{aligned}
x^2 - 2x - 2 > 0 &\quad \text{and} \quad \ln(x^2 - 2x - 2) \leq 0 \\
x^2 - 2x - 2 > 0 &\quad \text{and} \quad e^{\ln(x^2-2x-2)} \leq e^0 \\
x^2 - 2x - 2 > 0 &\quad \text{and} \quad x^2 - 2x - 2 \leq 1 \\
x^2 - 2x - 2 > 0 &\quad \text{and} \quad x^2 - 2x - 3 \leq 0 \\
[x - (1+\sqrt{3})][x - (1-\sqrt{3})] > 0 &\quad \text{and} \quad (x-3)(x+1) \leq 0. \qquad (1)
\end{aligned}$$

This inequality on the left was factored by finding the zeros of $x^2 - 2x - 2$:

$$x^2 - 2x - 2 = 0$$

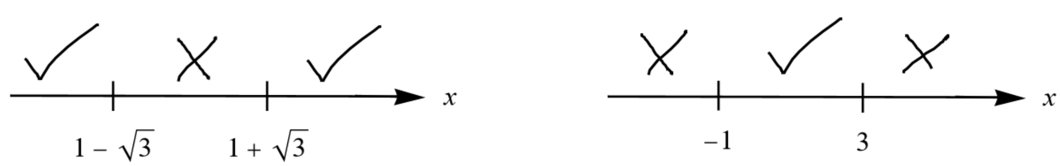
$$x = \frac{2 \pm \sqrt{4 - 4(-2)(1)}}{2}$$

$$x = \frac{2 \pm \sqrt{12}}{2}$$

$$x = \frac{2 \pm 2\sqrt{3}}{2}$$

$$x = 1 \pm \sqrt{3}.$$

The critical points for the inequality on the left are $1 - \sqrt{3} \approx -0.732$ and $1 + \sqrt{3} \approx 2.732$, and the critical points for the inequality on the right are -1 and 3. Partition two separate number lines at these points and test whether these inequalities are true or not in each of the intervals.



(1) then becomes

$$(x < 1 - \sqrt{3} \quad \text{or} \quad x > 1 + \sqrt{3}) \quad \text{and} \quad -1 \leq x \leq 3.$$

Therefore,

$$x \in \left[-1, 1 - \sqrt{3}\right) \cup \left(1 + \sqrt{3}, 3\right].$$

This is reflected in the graph below. If x takes on any value in this interval, the function is less than or equal to zero.